NONLINEAR COMMUTATORS FOR THE FRACTIONAL p-LAPLACIAN AND APPLICATIONS

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Abstract. We prove a nonlocal, nonlinear commutator estimate concerning the transfer of derivatives onto testfunctions. For the fractional p-Laplace operator it implies that solutions to certain degenerate nonlocal equations are higher differentiable. Also, weak fractional p-harmonic functions which a priori are less regular than variational solutions are in fact classical. As an application we show that sequences of uniformly bounded $\frac{n}{s}$-harmonic maps converge strongly outside at most finitely many points.

1. Introduction

The fractional p-Laplacian of order $s \in (0,1)$ on a domain $\Omega \subset \mathbb{R}^n$, $(-\Delta)^s_{p,\Omega} u$ is a distribution given by

$$(-\Delta)^s_{p,\Omega} u[\varphi] := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\ (\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\, dx\, dy$$

for $\varphi \in C^\infty_c(\Omega)$. It appears as the first variation of the $\dot{W}^{s,p}$-Sobolev norm

$$[u]^p_{W^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\ dy.$$

In this sense it is related to the classical p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

which appears as first variation of the $\dot{W}^{1,p}$-Sobolev norm $\|\nabla u\|^p_p$.

If $p = 2$ the fractional p-Laplacian on $\mathbb{R}^n$ becomes the usual fractional Laplace operator

$$(-\Delta)^s f = \mathcal{F}^{-1}(c\ |\xi|^{2s} \mathcal{F} f),$$

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where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. As a distribution

$$(-\Delta)^s f[\varphi] = \int_{\mathbb{R}^n} (-\Delta)^s f \varphi.$$

For an overview on the fractional Laplacian and fractional Sobolev spaces we refer to, e.g., [11, 4].

Due to the degeneracy for $p \neq 2$, regularity theory for equations involving the p-Laplacian is quite delicate, for example p-harmonic functions may not be C^2. The fractional p-Laplacian has recently received quite some interest, for example we refer to [2, 9, 10, 21, 18, 16, 13, 17, 23]. Higher regularity is one interesting and very challenging question where only very partial results are known, e.g. in [2] they obtain for $s \approx 1$ estimates in $C^{1,\alpha}$.

Our first result is a nonlinear commutator estimate for the fractional p-Laplacian. It measures how and at what price one can "transfer" derivatives to the testfunction. In the linear case $p = 2$ this is just integration by parts: Let c be the constant depending on s and ε so that $(-\Delta)^{s+\varepsilon} = c(-\Delta)^{\varepsilon} \circ (-\Delta)^s$. Then for any testfunction φ,

$$(-\Delta)^{s+\varepsilon} u[\varphi] = c(-\Delta)^s u[(-\Delta)^{\varepsilon}\varphi]$$

In the nonlinear case $p \neq 2$ (we shall restrict our attention for technical simplicity to $p \geq 2$) this is not true anymore. Instead we have

Theorem 1.1. *Let $s \in (0,1)$, $p \in [2,\infty)$ and $\varepsilon \in [0, 1-s)$. Take $B \subset \mathbb{R}^n$ a ball or all of $\mathbb{R}^n$. Let $u \in W^{s,p}(B)$ and $\varphi \in C_c^{\infty}(B)$. For a certain constant c depending on s, ε, p, denote the nonlinear commutator*

$$R(u,\varphi) := (-\Delta)^{s+\varepsilon}_{p,B} u[\varphi] - c(-\Delta)^s_{p,B} u[(-\Delta)^{\frac{\varepsilon p}{2}}\varphi].$$

Then we have the estimate

$$|R(u,\varphi)| \leq C\ \varepsilon\ [u]^{p-1}_{W^{s+\varepsilon,p}(B)} [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)}.$$

The fact that the ε appears in the estimate of $R(u,\varphi)$ is the main point in Theorem 1.1. It relies on a logarithmic potential estimate:

Lemma 1.2. *Let for $\alpha, \beta \in (0,n)$,*

$$k(x,y,z) = \left(|x-z|^{\alpha-n} \log \frac{|x-z|}{|x-y|} - |y-z|^{\alpha-n} \log \frac{|y-z|}{|x-y|} \right).$$

Let $\gamma \in (0,1)$, $p \in (1,\infty)$ and assume that $s := \gamma + \beta - \alpha \in (0,1)$. We consider the following semi-norm expression for $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$A(\varphi) := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} k(x,y,z)\ (-\Delta)^{\frac{\beta}{2}} \varphi(z)\ dz \right|^p \frac{dx\ dy}{|x-y|^{n+\gamma p}} \right)^{\frac{1}{p}}.$$

We have

$$A(\varphi) \leq C[\varphi]_{W^{s,p}(\mathbb{R}^n)}.$$

The additional factor ε in Theorem 1.1 facilitates estimates "close to the differential order s". More precisely

Theorem 1.3. *Let $s \in (0,1)$, $p \in [2,\infty)$, and a domain $\Omega \subset \mathbb{R}^n$, and $u \in W^{s,p}(\Omega)$ be a solution to $(-\Delta)^s_{p,\Omega} u = f$, i.e.*

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\ (\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\ dx\ dy = f[\varphi]$$

for all $\varphi \in C_c^\infty(\Omega)$. Then there is an $\varepsilon_0 > 0$ only depending on s, p, and Ω, so that for $\varepsilon \in (0,\varepsilon_0)$ the following holds: If $f \in (W^{s-\varepsilon(p-1),p}(\Omega))^*$ then $u \in W^{s+\varepsilon,p}_{loc}(\Omega)$.

More precisely, we have for any $\Omega_1 \Subset \Omega$ a constant $C = C(\Omega_1, \Omega, s, p)$ so that

$$[u]_{W^{s+\varepsilon,p}(\Omega_1)} \leq C\ \|f\|_{(W_0^{s-\varepsilon(p-1),p}(\Omega))^*} + C[u]_{W^{s,p}(\Omega)}.$$

Also, by Sobolev imbedding, the higher differentiability $W^{s+\varepsilon,p}_{loc}$ implies higher integrability i.e. $W^{s,p+\frac{pn}{n-\varepsilon p}}_{loc}$-estimates.

In the regime $p = 2$, a higher differentiability result similar to Theorem 1.3 was proven by Kuusi, Mingione, and Sire [18]. It seems also possible to extend their approach to the case $p > 2$. Their argument is based on a generalization of Gehring's Lemma and it is also valid for nonlinear versions, see [16]. Our method is similarly robust. Indeed one can show

Theorem 1.4. *Let $s \in (0,1)$, $p \in [2,\infty)$, and a domain $\Omega \subset \mathbb{R}^n$. Let $\phi : \mathbb{R} \to \mathbb{R}$ and $K(x,y)$ be a measurable kernel so that for some $C > 1$,*

$$|\phi(t)| \leq C|t|^{p-1}, \quad \phi(t)t \geq |t|^p \quad \forall t \in \mathbb{R},$$

and

$$C^{-1}|x-y|^{-n-sp} \leq K(x,y) \leq C|x-y|^{-n-sp}.$$

We consider for $u \in W^{s,p}(\Omega)$, the distribution $\mathcal{L}_{\phi,K,\Omega}(u)$

$$\mathcal{L}_{\phi,K,\Omega}(u)[\varphi] := \int_\Omega \int_\Omega K(x,y)\ \phi(u(x)-u(y))\ (\varphi(x)-\varphi(y))\ dx\ dy$$

Then the conclusions of Theorem 1.3 still hold if the fractional p-Laplace $(-\Delta)^s_{p,\Omega}$ is replaced with $\mathcal{L}_{\phi,K,\Omega}$.

Since the arguments for Theorem 1.4 follow closely the proof of Theorem 1.3, we leave this as an exercise to the interested reader.

There is also a reminiscent result to Theorem 1.3 the usual p-Laplace: A nonlinear potential estimate due to Iwaniec [14]. It implies that for u with $\operatorname{supp} u \subset \Omega$ there are maps v, R, so that

$$|\nabla u|^\varepsilon \nabla u = \nabla v + R,$$

with $\|\nabla v\|_{\frac{q}{1+\varepsilon},\Omega} \precsim \|\nabla u\|^{1+\varepsilon}_{q,\Omega}$ for all q and

$$\|R\|_{\frac{p+\varepsilon}{1+\varepsilon},\Omega} \precsim \varepsilon \|\nabla u\|^{1+\varepsilon}_{p+\varepsilon,\Omega}.$$

In this situation, the additional ε in the last estimate allows for estimates "close to the *integrability* order p". Indeed

$$\|\nabla u\|^{p+\varepsilon}_{p+\varepsilon,\Omega} = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |\nabla u|^{p-2}\nabla u R,$$

and thus,

$$\|\nabla u\|^{p+\varepsilon}_{p+\varepsilon,\Omega} \precsim |\Delta_p u[v]| + \varepsilon \|\nabla u\|^{p-1}_{p+\varepsilon,\Omega}\ \|\nabla u\|^{1+\varepsilon}_{p+\varepsilon,\Omega}.$$

In particular, if ε is small enough and $\Delta_p u$ is in $(W_0^{1,\frac{p+\varepsilon}{1+\varepsilon}}(\Omega))^*$, then $u \in W^{1,p+\varepsilon}(\Omega)$.

The commutator estimate in Theorem 1.1 also allows to estimate very weak solutions - i.e. solutions whose initial regularity assumptions are below the variationally natural regularity:

In the local regime, the distributional p-Laplacian $\Delta_p u[\varphi]$ is well defined for $\varphi \in C_c^\infty(\Omega)$ whenever $u \in W^{1,p-1}_{loc}(\Omega)$. The variationally natural regularity assumption is however $W^{1,p}$, since Δ_p appears as first variation of $\|\nabla u\|^p_{p,\Omega}$. For the p-Laplacian, Iwaniec and Sbordone [15] showed that some weak p-harmonic functions are in fact classical variational solutions:

Theorem 1.5 (Iwaniec-Sbordone)**.** *For any $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$, there are exponents $1 < r_1 < p < r_2 < \infty$ so that every (weakly) p-harmonic map,*

$$\Delta_p u = 0,$$

satisfying $u \in W^{1,r_1}_{loc}(\Omega)$ *indeed belongs to* $W^{1,r_2}_{loc}(\Omega)$.

Again, while the p-Laplace improves its solution's *integrability*, the fractional p-Laplace improves its solution's *differentiability*. The distributional fractional p-Laplace $(-\Delta)^s_{p,\Omega} u[\varphi]$ is well defined for $\varphi \in C_c^\infty(\Omega)$ whenever $u \in W^{q,p-1}(\Omega)$ for any $q > 0$ with $q \geq (\frac{sp-1}{p-1})_+$. We have

Theorem 1.6. *For any* $s \in (0,1)$ $p \in (2,\infty)$, $\Omega \subset \mathbb{R}^n$, *there are exponents* $1 < r_1 < p < r_2 < \infty$ *and* $t_1 < s < t_2$ *so that every (weakly) s-p-harmonic map,*

$$(-\Delta)^s_{p,\Omega} u = 0,$$

satisfying $u \in W^{t_1,r_1}(\Omega)$ *indeed belongs to* $W^{t_2,r_2}_{loc}(\Omega)$.

The arguments for Theorem 1.6 are quite similar to the ones in Theorem 1.3, and we shall skip them.

Let us state an important application of Theorem 1.3: It is concerning degenerate fractional harmonic maps into spheres $\mathbb{S}^N \subset \mathbb{R}^{N+1}$: In [21] we proved that for $s \in (0,1)$ critical points of the energy

$$\mathcal{E}_s(u) := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+s\frac{n}{s}}}\, dx\, dy, \quad u : \Omega \subset \mathbb{R}^n \to \mathbb{S}^N$$

are Hölder continuous. Indeed, together with Theorem 1.3 the estimates in [21] impAly a sharper result

Theorem 1.7 (ε-regularity for fractional harmonic maps)**.** *For any open set* $\Omega \subset \mathbb{R}^n$ *there is a* $\delta > 0$ *so that for any* $\Lambda > 0$ *there exists* $\varepsilon > 0$ *and the following holds: Let* $u \in W^{s,\frac{n}{s}}(\Omega, \mathbb{S}^N)$ *with*

$$[u]_{W^{s,\frac{n}{s}}(\Omega)} \leq \Lambda \tag{1.1}$$

be a critical point of $\mathcal{E}_s(u)$, *i.e.*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}_s\left(\frac{u+t\varphi}{|u+t\varphi|}\right) = 0 \quad \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{1.2}$$

If on a ball $2B \subset \Omega$ *we have*

$$[u]_{W^{s,\frac{n}{s}}(2B)} \leq \varepsilon, \tag{1.3}$$

then on the ball B *(the ball concentric to* $2B$ *with half the radius),*

$$[u]_{W^{s+\delta,\frac{n}{s}}(B)} \leq C_{\Lambda,B}.$$

This kind of ε-regularity estimate is crucial for compactness and bubble analysis for fractional harmonic maps. Da Lio obtained quantization results [6] for $n = 1$ and $s = \frac{1}{2}$. With the help of Theorem 1.7 one can extend her *compactness* estimates to all $s \in (0,1)$, $n \in \mathbb{N}$. More precisely, we have the following result extending the first part of [6, Theorem 1.1].

Theorem 1.8. *Let $u_k \in \dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{S}^{N-1})$ be a sequence of $(s, \frac{n}{s})$-harmonic maps in the sense of* (1.2) *such that*

$$[u_k]_{W^{s,\frac{n}{s}}(\mathbb{R}^n,\mathbb{S}^{N-1})} \leq C.$$

Then there is $u_\infty \in \dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{S}^{N-1})$ and a possibly empty set $\{\alpha_1, \ldots, \alpha_l\}$ such that up to a subsequence we have strong convergence away from $\{\alpha_1, \ldots, \alpha_l\}$, that is

$$u_k \xrightarrow{k\to\infty} u_\infty \quad \text{in } W^{s,\frac{n}{s}}_{loc}(\mathbb{R}^n \backslash \{\alpha_1, \ldots, \alpha_l\}).$$

A more precise analysis of compactness and the formation of bubbles will be part of a future work.

2. Outline and Notation

In Section 3 we will prove the commutator estimate, Theorem 1.1. Roughly speaking, we compute the kernel $\kappa_\varepsilon(x, y, z)$ of the commutator and show that its derivative *in* ε (which gives a logarithmic potential) induces a bounded operator. The latter estimate is contained in Lemma 1.2 which we shall prove via Littlewood-Paley theory in Section 4.

Based on Theorem 1.1 we will then proceed in Section 5 with the proof of Theorem 1.3. Finally, the consequences of this analysis, i.e. higher differentiability result for p-fractional harmonic maps is sketched in Section 6, and the proof of Theorem 1.8 in Session 7. In the appendix we record a few necessary tools used throughout the proofs.

We try to keep the notation as simple as possible. For a ball B, λB denotes the concentric ball with λ-times the radius. With

$$(u)_B := |B|^{-1} \int_B u$$

we denote the mean value.

The dual norm of the p-Laplacian is denoted as

$$\|(-\Delta)^s_{p,\Omega}u\|_{(W^{t,p}_0(\Omega))^*} \equiv \sup_{\varphi} |(-\Delta)^s_{p,\Omega}u[\varphi]|$$

where the supremum is taken over $\varphi \in C_c^\infty(\Omega)$ with $[\varphi]_{W^{t,p}(\mathbb{R}^n)} \leq 1$.

We already defined the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$. Its inverse I^s is the Riesz potential, which for some constant $c \in \mathbb{R}$ can be written as

$$I^s g(x) = c\int_{\mathbb{R}^n} |x-z|^{s-n} g(z)\, dz. \tag{2.1}$$

In the estimates, the constants can change from line to line. Whenever we deem the constant unimportant to the argument, we will drop it, writing $A \precsim B$ if $A \leq C \cdot B$ for some constant $C > 0$. Similarly we will use $A \approx B$ whenever A and B are comparable.

3. The commutator estimate: Proof of Theorem 1.1

Proof. Recall that for $t \in (0,n)$ there is a constant $c \in \mathbb{R}$ so that for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$c\int_{\mathbb{R}^n} |x-z|^{t-n} (-\Delta)^{\frac{t}{2}}\varphi(z)\, dz = I^t(-\Delta)^{\frac{t}{2}}\varphi(x) = \varphi(x). \tag{3.1}$$

We write

$$(-\Delta)^{s+\varepsilon}_{p,B}u[\varphi] = \int_B\int_B \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\frac{\varphi(x)-\varphi(y)}{|x-y|^{\varepsilon p}})}{|x-y|^{n+sp}}\, dx\, dy$$

$$\overset{(3.1)}{=} \int_{\mathbb{R}^n}\int_B\int_B \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\frac{|x-z|^{t+\varepsilon p-n}-|y-z|^{t+\varepsilon p-n}}{|x-y|^{\varepsilon p}})}{|x-y|^{n+sp}}\, dx\, dy\, (-\Delta)^{\frac{t+\varepsilon p}{2}}\varphi(z)dz$$

$$= \int_{\mathbb{R}^n}\int_B\int_B \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\,(|x-z|^{t-n}-|x-y|^{t-n}))}{|x-y|^{n+sp}}\, dx\, dy\, (-\Delta)^{\frac{t+\varepsilon p}{2}}\varphi(z)dz$$

$$+ \int_{\mathbb{R}^n}\int_B\int_B \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\kappa_\varepsilon(x,y,z)}{|x-y|^{n+sp}}\, dx\, dy\, (-\Delta)^{\frac{t+\varepsilon p}{2}}\varphi(z)dz$$

with

$$\kappa_\varepsilon(x,y,z) := \left(\frac{|x-z|^{t+\varepsilon p-n}-|y-z|^{t+\varepsilon p-n}}{|x-y|^{\varepsilon p}}\right) - (|x-z|^{t-n}-|x-y|^{t-n}).$$

Using again (3.1) this reads as

$$R(u,\varphi) := (-\Delta)^{s+\varepsilon}_{p,B}u[\varphi] - c(-\Delta)^s_{p,B}u[(-\Delta)^{\frac{\varepsilon p}{2}}\varphi]$$

$$= \int\limits_{\mathbb{R}^n} \int\limits_B \int\limits_B \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\kappa_\varepsilon(x,y,z)}{|x-y|^{n+sp}}\, dx\, dy\, (-\Delta)^{\frac{t+\varepsilon p}{2}} \varphi(z) dz.$$

Since $\kappa_0(x,y,z) = 0$ for almost all $x,y,z \in \mathbb{R}^n$,

$$\kappa_\varepsilon(x,y,z) = \int_0^\varepsilon \frac{d}{d\delta} \kappa_\delta(x,y,z)\ d\delta.$$

We thus set

$$\begin{aligned} k_\delta(x,y,z) &:= |x-y|^{\delta p} \frac{d}{d\delta} \kappa_\delta(x,y,z) \\ &= \left(|x-z|^{t+\delta p - n} \log \frac{|x-z|}{|x-y|} - |y-z|^{t+\delta p-n} \log \frac{|y-z|}{|x-y|} \right) \end{aligned}$$

and arrive at $R(u,\varphi)$ being equal to

$$\int\limits_0^\varepsilon \int\limits_B \int\limits_B \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{(s+\varepsilon)(p-1)}} \left(\int\limits_{\mathbb{R}^n} \frac{\kappa_\delta(x,y,z)\ (-\Delta)^{\frac{t+\varepsilon p}{2}} \varphi(z) dz}{|x-y|^{s+\varepsilon-(\varepsilon-\delta)p}} \right) \frac{dx\ dy\ d\delta}{|x-y|^n}.$$

With Hölder inequality we get the upper bound for $|R(u,\varphi)|$

$$\varepsilon [u]^{p-1}_{W^{s+\varepsilon,p}(B)} \sup_{\delta \in (0,\varepsilon)} \left(\int\limits_B \int\limits_B \left(\int\limits_{\mathbb{R}^n} \frac{\kappa_\delta(x,y,z)\ (-\Delta)^{\frac{t+\varepsilon p}{2}} \varphi(z) dz}{|x-y|^{s+\varepsilon-(\varepsilon-\delta)p}} \right)^p \frac{dx\ dy}{|x-y|^n} \right)^{\frac{1}{p}}.$$

This falls into the realm of Lemma 1.2, for

$\alpha := t + \delta p, \quad \beta := t + \varepsilon p, \quad \gamma := s + \varepsilon - (\varepsilon - \delta)p, \quad \gamma + \beta - \alpha = s + \varepsilon.$

This concludes the proof. □

4. Logarithmic potential estimate: Proof of Lemma 1.2

For the proof of Lemma 1.2 we will use the Littlewood-Paley decomposition: We refer to the Triebel monographs, e.g. [22] and [12] for a complete picture of this tool. We will only need few properties:

For a tempered distribution f we define f_j to be the Littlewood-Paley projections $f_j := P_j f$, where

$$P_j f(x) := \int\limits_{\mathbb{R}^n} 2^{jn} p(2^j(x-z)) f(z)\ dz.$$

Here, p is a Schwartz function, and it can be chosen in a way such that

$$\sum_{j \in \mathbb{Z}} f_j = f \quad \text{for all } f \in \mathcal{S}'. \tag{4.1}$$

For any $j \in \mathbb{Z}$ we have the estimate for Riesz potentials and derivatives (cf. (2.1))

$$\|I^s|(-\Delta)^{\frac{t}{2}} f_j|\|_p \precsim \sum_{i=j-1}^{j+1} 2^{j(t-s)} \|f_i\|_p \tag{4.2}$$

The homogeneous semi-norm for the Triebel space $\dot{F}^s_{p,p} = \dot{B}^s_{p,p}$ is

$$\|f\|_{\dot{F}^s_{p,p}} := \left(\sum_{j\in\mathbb{Z}} 2^{jsp} \|f_j\|_p^p \right)^{\frac{1}{p}}. \tag{4.3}$$

Crucially to us, the Triebel spaces are equivalent to Sobolev spaces: For $s \in (0,1)$ we have the identification

$$\|f\|_{\dot{F}^s_{p,p}} \approx [f]_{W^{s,p}(\mathbb{R}^n)}. \tag{4.4}$$

Proof of Lemma 1.2. For $k \in \mathbb{Z}$, we use the annular cutoff function

$$\chi_{|y|\approx 2^{-k}} := \chi_{B_{2^{-k}(0)} \setminus B_{2^{-k-1}(0)}}(y).$$

With this and (4.1), setting

$$T\varphi(x,y) := \int_{\mathbb{R}^n} k(x,y,z)\ (-\Delta)^{\frac{\beta}{2}} \varphi(z)\ dz,$$

we decompose

$$A(\varphi)^p \precsim \sum_{k\in\mathbb{Z}, j\in\mathbb{Z}} I_{j,k},$$

where

$$I_{j,k} := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{|x-y|\approx 2^{-k}}\, |T\varphi(x,y)|^{p-1}\ \ |T\varphi_j(x,y)|\ \frac{dx\ dy}{|x-y|^{n+\gamma p}}.$$

Set

$$a_k := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{|x-y|\approx 2^{-k}}\, |T\varphi(x,y)|^p\, \frac{dx\ dy}{|x-y|^{n+\gamma p}} \right)^{\frac{1}{p}}$$

and

$$b_j := 2^{j(\gamma+\beta-\alpha)}\ \|\varphi_j\|_p.$$

Note that with (4.3) and (4.4)

$$\left(\sum_{k\in\mathbb{Z}} a_k^p \right)^{\frac{1}{p}} \approx A(\varphi) \quad \text{and} \quad \left(\sum_{j\in\mathbb{Z}} b_j^p \right)^{\frac{1}{p}} \approx \|\varphi\|_{\dot{F}^s_{p,p}} \approx [\varphi]_{W^{s,p}(\mathbb{R}^n)}. \tag{4.5}$$

Then with Hölder inequality,

$$I_{j,k} \precsim a_k^{p-1} \left(\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \chi_{|x-y|\approx 2^{-k}} \, |T\varphi_j(x,y)|^p \quad \frac{dx\ dy}{|x-y|^{n+\gamma p}} \right)^{\frac{1}{p}}$$
$$=: a_k^{p-1}\ \tilde{I}_{j,k}.$$

Now we have to possibilities of estimating $\tilde{I}_{j,k}$:

Firstly, for any small $\sigma \in (0,\alpha)$ we can employ the estimate $|\log \frac{|x-z|}{|x-y|}| \precsim \frac{|x-y|^\sigma}{|x-z|^\sigma} + \frac{|x-z|^\sigma}{|x-y|^\sigma}$, and have an estimate with Riesz potentials (2.1)

$$\int\limits_{\mathbb{R}^n} |x-z|^{\alpha-n} \log \frac{|x-z|}{|x-y|} |(-\Delta)^{\frac{\beta}{2}} \varphi_j(z)|\ dz$$
$$\precsim |x-y|^{-\sigma} I^{\alpha+\sigma} |(-\Delta)^{\frac{\beta}{2}} \varphi_j|(x) + |x-y|^{\sigma} I^{\alpha-\sigma} |(-\Delta)^{\frac{\beta}{2}} \varphi_j|(x).$$

Having in mind (4.2) we obtain the estimate

$$\tilde{I}_{j,k} \precsim 2^{k(\frac{n+\gamma p}{p})}\ 2^{k\sigma} 2^{-k\frac{n}{p}} \|I^{\alpha+\sigma}|(-\Delta)^{\frac{\beta}{2}}\varphi_j|\|_p + 2^{k(\frac{n+\gamma p}{p})}\ 2^{-k\sigma}2^{-k\frac{n}{p}} \|I^{\alpha-\sigma}|(-\Delta)^{\frac{\beta}{2}}\varphi_j|\|_p$$
$$\precsim 2^{(k-j)(\gamma+\sigma)}(b_{j-1}+b_j+b_{j+1}) + 2^{(k-j)(\gamma-\sigma)}(b_{j-1}+b_j+b_{j+1}).$$

This is our first estimate:

$$\tilde{I}_{j,k} \precsim 2^{(k-j)(\gamma-\sigma)}\ (2^{2\sigma(k-j)}+1)\ (b_{j-1}+b_j+b_{j+1}). \tag{4.6}$$

Secondly, by a substitution we can write

$$T\varphi_j(x,y) = \int_{\mathbb{R}^n} |z|^{\alpha-n} \log \frac{|z|}{|x-y|} \left((-\Delta)^{\frac{\beta}{2}} \varphi_j(z+x) - (-\Delta)^{\frac{\beta}{2}} \varphi_j(z+y) \right) dz.$$

We use now $|f(x)-f(y)| \precsim |x-y|(\mathcal{M}|\nabla f|(x) + \mathcal{M}|\nabla f|(y))$, where $\mathcal{M}$ is the Hardy-Littlewood maximal function. Then, again for any $\sigma > 0$,

$$|T\varphi_j(x,y)|$$
$$\precsim |x-y| \int_{\mathbb{R}^n} |z|^{\alpha-n} \left| \log \frac{|z|}{|x-y|} \right|\ \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(z+x)\ dz$$
$$+ |x-y| \int_{\mathbb{R}^n} |z|^{\alpha-n} \left| \log \frac{|z|}{|x-y|} \right|\ |\mathcal{M}(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(z+x)\ dz$$
$$\precsim |x-y|^{1-\sigma} I^{\alpha+\sigma} \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(x)$$
$$+ |x-y|^{1-\sigma} I^{\alpha+\sigma} \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(y)$$
$$+ |x-y|^{1+\sigma} I^{\alpha-\sigma} \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(x)$$
$$+ |x-y|^{1+\sigma} I^{\alpha-\sigma} \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(y)$$

Consequently, our second estimate is

$$\begin{aligned}\tilde{I}_{j,k} \precsim& 2^{k(\gamma-1+\sigma)}\|I^{\alpha+\sigma}\mathcal{M}|(-\Delta)^{\frac{\beta}{2}}\nabla\varphi_j\|_p + 2^{k(\gamma-1-\sigma)}\|I^{\alpha-\sigma}\mathcal{M}|(-\Delta)^{\frac{\beta}{2}}\nabla\varphi_j\|_p\\ \precsim& 2^{k(\gamma-1+\sigma)}\ 2^{j(-\alpha-\sigma+\beta+1)}\|\varphi_j\|_p + 2^{k(\gamma-1-\sigma)}\ 2^{j(-\alpha+\sigma+\beta+1)}\|\varphi_j\|_p.\end{aligned}$$

Together with (4.6) we thus have

$$\tilde{I}_{k,j} \precsim \min\{2^{(k-j)(\gamma-\sigma)}\ (2^{2\sigma(k-j)}+1), 2^{(j-k)(1-\gamma-\sigma)}\ (1+2^{(j-k)(2\sigma)})\}\ (b_{j-1}+b_j+b_{j+1}).$$

In particular, since $\gamma \in (0,1)$ pick any $0 < \sigma < \min\{\gamma, 1-\gamma\}$ – which, as we shall see in a moment, makes the following sums convergent:

$$\begin{aligned}A(\varphi)^p \precsim& \sum_{j\in\mathbb{Z}}\sum_{k=j+1}^{\infty} 2^{(j-k)(1-\gamma-\sigma)}\ (b_{j-1}+b_j+b_{j+1})\ a_j^{p-1}\\ &+\sum_{j\in\mathbb{Z}}\sum_{k=-\infty}^{j-1} 2^{(k-j)(\gamma-\sigma)}\ (b_{j-1}+b_j+b_{j+1})\ a_j^{p-1}\\ &+\sum_{j\in\mathbb{Z}}(b_{j-1}+b_j+b_{j+1})\ a_j^{p-1}\\ =:& I + II + III.\end{aligned}$$

With Hölder inequality and (4.5),

$$III \precsim (\sum_{j\in\mathbb{Z}} b_j^p)^{\frac{1}{p}}\ (\sum_{j\in\mathbb{Z}} a_j^p)^{\frac{p-1}{p}} = A(\varphi)^{p-1}\ [\varphi]_{W^{s,p}(\mathbb{R}^n)}.$$

As for I, for any $\varepsilon > 0$,

$$\begin{aligned}I =& \sum_{j\in\mathbb{Z}}\sum_{k=j}^{\infty} 2^{(j-k)(1-\gamma-\sigma)}\ b_j\ a_k^{p-1}\\ \precsim& \sum_{j\in\mathbb{Z}}\sum_{k=j}^{\infty} 2^{(j-k)(1-\gamma-\sigma)}\ (\varepsilon^p b_j^p\ + \varepsilon^{-p'} a_k^p)\\ =& C_{1-\gamma-\sigma}\varepsilon^p \sum_{j\in\mathbb{Z}} b_j^p\ + \varepsilon^{-p'}\sum_{j\in\mathbb{Z}}\sum_{k=j}^{\infty} 2^{(j-k)(1-\gamma-\sigma)} a_k^p\\ =& C_{1-\gamma-\sigma}\varepsilon^p \sum_{j\in\mathbb{Z}} b_j^p\ + \varepsilon^{-p'}\sum_{k\in\mathbb{Z}}\sum_{j=-\infty}^{k} 2^{(j-k)(1-\gamma-\sigma)} a_k^p\\ =& C_{1-\gamma-\sigma}\varepsilon^p \sum_{j\in\mathbb{Z}} b_j^p\ + \varepsilon^{-p'} C_{1-\gamma-\sigma}\sum_{k\in\mathbb{Z}} a_k^p\\ \approx& \varepsilon^p [\varphi]^p_{W^{s,p}(\mathbb{R}^n)}\ + \varepsilon^{-p'} C_{1-\gamma-\sigma} A(\varphi)^p\end{aligned}$$

The same works for II:

$$II = \sum_{j\in\mathbb{Z}} \sum_{k=-\infty}^{j-1} 2^{(k-j)(\gamma-\sigma)}\ b_j\ a_k^{p-1}$$
$$\precsim \varepsilon^p [\varphi]^p_{W^{s,p}(\mathbb{R}^n)} + \varepsilon^{-p'} C_{1-\gamma-\sigma} A(\varphi)^p$$

Together,

$$I + II \precsim \varepsilon^p [\varphi]^p_{W^{s,p}(\mathbb{R}^n)} + \varepsilon^{-p'} C_{1-\gamma-\sigma} A(\varphi)^p,$$

which holds for any $\varepsilon > 0$. Pick

$$\varepsilon := [\varphi]^{-\frac{1}{p'}}_{W^{s,p}(\mathbb{R}^n)}\ A(\varphi)^{\frac{1}{p'}}.$$

Then

$$A(\varphi)^p \leq I + II + III \precsim A(\varphi)^{p-1}\ [\varphi]_{W^{s,p}(\mathbb{R}^n)}.$$

We conclude dividing both sides by $A(\varphi)^{p-1}$. $\square$

5. Higher Differentiability: Proof of Theorem 1.3

In view of Lemma A.1 we can assume w.l.o.g. that Ω is a bounded open set, and that the support of u is strictly contained in some open set $\Omega_1 \Subset \Omega$. Then Theorem 1.3 follows from

Lemma 5.1. *Let $\Omega_1 \Subset \Omega$ two open, bounded sets, $s \in (0,1)$, $p \in [2,\infty)$. Then there exists an $\varepsilon_0 > 0$ so that for any $\varepsilon \in (0,\varepsilon_0)$,*

$$[u]^{p-1}_{W^{s+\varepsilon,p}(\Omega)} \precsim [u]^{p-1}_{W^{s,p}(\Omega)} + \|(-\Delta)^s_{p,\Omega} u\|_{(W_0^{s-\varepsilon(p-1),p}(\Omega))^*}.$$

Proof. We can find finitely many balls $(B_k)_{k=1}^K \subset \Omega$ so that $\bigcup_{k=1}^N B_k \supset \Omega_1$. We denote with $10B_k$ the concentric balls with ten times the radius, and may assume $\bigcup_{k=1}^N 10B_k \subset \Omega$.

Denote

$$\Gamma_s := [u]^p_{W^{s,p}(\Omega)}, \quad \Gamma_{s+\varepsilon} := [u]^p_{W^{s+\varepsilon,p}(\Omega)}.$$

We then have

$$\Gamma_{s+\varepsilon} \precsim \sum_{k=1}^K [u]^p_{W^{s+\varepsilon,p}(2B_k)} + \sum_{k=1}^K \int_{\Omega\backslash 2B_k} \int_{B_k} \frac{|u(x)-u(y)|^p}{|x-y|^{n+(s+\varepsilon)p}}\ dx\ dy.$$

As for the second term, because of the disjoint support of the integrals we find

$$\int_{\Omega\backslash 2B_k} \int_{B_k} \frac{|u(x)-u(y)|^p}{|x-y|^{n+(s+\varepsilon)p}}\ dx\ dy \precsim (\operatorname{diam} B_k)^{-\varepsilon p}\ \Gamma_s.$$

That is

$$\Gamma_{s+\varepsilon} \precsim \sum_{k=1}^{K} [u]^p_{W^{s+\varepsilon,p}(2B_k)} + \Gamma_s.$$

With Lemma A.2 and Poincaré inequality, Proposition A.3, for any $\delta > 0$,

$$\Gamma_{s+\varepsilon} \precsim \delta^p \Gamma_{s+\varepsilon} + C_\delta \Gamma_s + \sum_{k=1}^{K} \delta^{-p'} \left(\sup_{\varphi} (-\Delta)^{s+\varepsilon}_{p,8B_k} u[\varphi] \right)^{\frac{p}{p-1}}$$

where the supremum is over all $\varphi \in C^\infty_c(4B_k)$ and $[\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1$. Here we also used that $\bigcup_{k=1}^K 8B_k$ covers no more than Ω. Choosing δ sufficiently small, we can estimate $\Gamma_{s+\varepsilon}$ by

$$\Gamma_s + \sum_{k=1}^{K} \left(\sup \left\{ |(-\Delta)^{s+\varepsilon}_{p,8B_k} u[\varphi]| : \ \varphi \in C^\infty_c(4B_k), [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1 \right\} \right)^{\frac{p}{p-1}}.$$

With Theorem 1.1 this can be estimated by

$$\Gamma_s + \varepsilon^{\frac{p}{p-1}} \Gamma_{s+\varepsilon}$$

$$+ \sum_{k=1}^{K} \left(\sup \left\{ |(-\Delta)^{s}_{p,8B_k} u[(-\Delta)^{\frac{\varepsilon p}{2}} \varphi]| : \ \varphi \in C^\infty_c(4B_k), [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1 \right\} \right)^{\frac{p}{p-1}}.$$

If $\varepsilon \in [0, \varepsilon_0)$ for ε_0 small enough, we can again absorb $\Gamma_{s+\varepsilon}$. The estimate for $\Gamma_{s+\varepsilon}$ becomes

$$\Gamma_s + \sum_{k=1}^{K} \left(\sup \left\{ |(-\Delta)^{s}_{p,8B_k} u[(-\Delta)^{\frac{\varepsilon p}{2}} \varphi]| : \ \varphi \in C^\infty_c(4B_k), [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1 \right\} \right)^{\frac{p}{p-1}}.$$

Next, we need to transform $(-\Delta)^{\frac{\varepsilon p}{2}} \varphi$ into a feasible testfunction, and denoting the usual cutoff function with $\eta_{6B_k} \in C^\infty_c(6B_k)$, $\eta_{6B_k} \equiv 1$ in $5B_k$

$$(-\Delta)^{\frac{\varepsilon p}{2}} \varphi =: \psi + (1 - \eta_{6B_k})(-\Delta)^{\frac{\varepsilon p}{2}} \varphi$$

Then $\psi \in C^\infty_c(6B_k)$

$$[\psi]_{W^{s-\varepsilon(p-1),p}(\Omega)} \precsim C_k [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq C_k.$$

Moreover, the disjoint support of $(1 - \eta_{6B_k})$ and φ implies (see, e.g., [3, Lemma A.1])

$$[(1 - \eta_{6B_k})(-\Delta)^{\frac{\varepsilon p}{2}} \varphi]_{\mathrm{Lip}} \leq C_k \ [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)}.$$

Consequently,

$$|(-\Delta)^s_{p,8B_k} u[(-\Delta)^{\frac{\varepsilon p}{2}} \varphi - \psi]| \precsim [u]^{p-1}_{W^{s,p}(\Omega)}.$$

Hence, our estimate for $\Gamma_{s+\varepsilon}$ now looks like

$$\Gamma_s+\sum_{k=1}^{K}\left(\sup\left\{|(-\Delta)^s_{p,8B_k}u[\psi]|:\ \psi\in C_c^\infty(6B_k),[\psi]_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^n)}\leq 1\right\}\right)^{\frac{p}{p-1}}.$$

Finally, we need to transform the support of $(-\Delta)^{\frac{s}{2}}_p$ from $8B_k$ to Ω. Since $\operatorname{supp}\psi\subset 6B_k$, the disjoint support of the integrals gives

$$\begin{aligned}
&|(-\Delta)^s_{p,8B_k}u[\psi]-(-\Delta)^s_{p,\Omega}u[\psi]|\\
\precsim&\int_{\Omega\backslash 8B_k}\int_{7B_k}\frac{|u(x)-u(y)|^{p-1}\,|\psi(x)-\psi(y)|}{|x-y|^{n+sp}}\,dx\,dy\\
\leq& C_k[u]^{p-1}_{W^{s,p}(\Omega)}[\psi]_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^n)}.
\end{aligned}$$

This implies the final estimate of $\Gamma_{s+\varepsilon}$ by

$$\Gamma_s+\left(\sup\left\{|(-\Delta)^s_{p,\Omega}u[\psi]|:\ \psi\in C_c^\infty(\Omega),[\psi]_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^n)}\leq 1\right\}\right)^{\frac{p}{p-1}}.$$

□

6. Differentiability of p-harmonic maps: Proof of Theorem 1.7

For $B\subset\mathbb{R}^n$, $t\in(0,1)$, we set

$$T_{t,B}u(z)=\int_B\int_B\frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\,(|x-z|^{t-n}-|y-z|^{t-n})}{|x-y|^{n+s\frac{n}{s}}}\,dx\,dy.$$

$T_{t,B}u$ was introduced in [21] because of the following relation

(6.1)
$$\begin{aligned}
&c\int_{\mathbb{R}^n}T_{t,B}u(z)\ \varphi(z)\ dz\\
=&\int_B\int_B\frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\ (I^t\varphi(x)-I^t\varphi(y))}{|x-y|^{n+s\frac{n}{s}}}\ dx\ dy.
\end{aligned}$$

From [21, in particular (3.1), Lemma 3.3, 3.4, 3.5] we have the following

Theorem 6.1. *Let u satisfy* (1.1) *and* (1.2) *in an open set Ω. Assume that on the Ball $2B$ for a small enough $\varepsilon>0$ (depending on Λ)* (1.3) *holds. Then there is $t_0<s$, $\sigma>0$, so that for some $\gamma_2>\gamma_1\gg 1$ for any ball $B_{\gamma_2\rho}\subset B$*

$$[u]_{W^{s,\frac{n}{s}}(B_\rho)}\precsim C_\Lambda\rho^\sigma, \tag{6.2}$$

and

$$\|T_{t_0,B_{\gamma_1\rho}}u\|_{\frac{n}{n-t_0},B_\rho}\leq C_\Lambda\rho^\sigma. \tag{6.3}$$

Estimate (6.3) looks almost as if $T_{t_0,B_{\gamma_1\rho}}$ belongs locally to a Morrey space. But the domain dependence on $B_{\gamma_1\rho}$ bars us from exploiting this. The following proposition removes the domain dependence.

Proposition 6.2. *Under the assumptions of Theorem 6.1 there exists* $\gamma > 1$, $\sigma > 0$ *so that*

$$\|T_{t_0,B}u\|_{\frac{n}{n-t_0},B_\rho} \leq C_{B,\Lambda}\rho^\sigma$$

for any ball so that $B_{\gamma\rho} \subset B$.

Proof. Set $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq 1$ to be chosen later. Take $\gamma := 2\gamma_1$ with γ_1 from (6.3). We will always assume $\rho < 1$.

For some $\varphi \in C_c^\infty(B_{\rho^{\kappa_1}})$, $\|\varphi\|_{\frac{n}{t_0}} \leq 1$ we have

$$\begin{aligned}
&\|T_{t_0,B}u\|_{\frac{n}{n-t_0},B_{\rho^{\kappa_1}}}\\
\precsim &\int_{\mathbb{R}^n} T_{t_0,B}u\ \varphi\\
\overset{(6.1)}{\approx} &\int_B\int_B \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\ (I^{t_0}\varphi(x)-I^{t_0}\varphi(y))}{|x-y|^{n+s\frac{n}{s}}}\ dx\ dy.
\end{aligned}$$

We will now use several cutoffs to slice φ into the right form. This kind of arguments and the consequent (tedious) estimates have been used several times in work related to fractional harmonic maps, cf. e.g. [8, 7, 5, 3, 21, 19, 20], and we will not repeat them in detail. We will also assume that $\kappa_1 > \kappa_2 > \kappa_3$. If they are equal, to keep the "disjoint support estimates" working one needs to use cutoff functions on twice, four times etc. of the Balls.

For a cutoff function $\eta_{B_{\rho^{\kappa_2}}} \in C_c^\infty(B_{2\rho^{\kappa_2}})$, $\eta_{B_{\rho^{\kappa_2}}} \equiv 1$ on $B_{\rho^{\kappa_2}}$, we have

$$I^{t_0}\varphi := \psi + (1-\eta_{B_{\rho^{\kappa_2}}})I^{t_0}\varphi.$$

Note that $\psi \in C_c^\infty(B_{2\rho^{\kappa_2}})$ and[1]

$$\|(-\Delta)^{\frac{t_0}{2}}\psi\|_{\frac{n}{t_0}} + [\psi]_{W^{t_0,\frac{n}{t_0}}(\mathbb{R}^n)} \precsim \|\varphi\|_{\frac{n}{t_0}}. \tag{6.4}$$

The disjoint support of $(1-\eta)$ and φ ensures (see [3, Lemma A.1])

$$[I^{t_0}\varphi - \psi]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \precsim \rho^{(\kappa_1-\kappa_2)(n-t_0)}\ \|\varphi\|_{\frac{n}{t_0}}. \tag{6.5}$$

We furthermore decompose

$$(-\Delta)^{\frac{t_0}{2}}\psi =: \phi + (1-\eta_{B_{\rho^{\kappa_3}}})(-\Delta)^{\frac{t_0}{2}}\psi.$$

[1]This is true if $\frac{n}{t_0} \geq 2$, since then $[f]_{W^{t_0,\frac{n}{t_0}}} \leq \|(-\Delta)^{\frac{t_0}{2}}f\|_{\frac{n}{t_0}}$. If $\frac{n}{t_0} < 2$ one has to adapt the estimate, but the results remains true.

Then $\phi \in C_c^\infty(B_{2\rho^{\kappa_3}})$ and

$$\|\phi\|_{\frac{n}{t_0}} \precsim \|\varphi\|_{\frac{n}{t_0}}, \tag{6.6}$$

$$\|\nabla(\psi - I^{t_0}\phi)\|_\infty \precsim \rho^{-\kappa_3+(\kappa_2-\kappa_3)n}\ \|\varphi\|_{\frac{n}{t_0}}. \tag{6.7}$$

Again with (6.1), we then have

$$\|T_{t_0,B}u\|_{\frac{n}{n-t_0},B_\rho} \precsim |I| + |II| + |III| + |IV|$$

where

$$I := \int T_{t_0,B_{\gamma\rho}}u\ \phi$$

$$II := \int_{B_{\gamma\rho}}\int_{B_{\gamma\rho}} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\ ((\psi - I^{t_0}\phi)(x) - (\psi - I^{t_0}\phi)(y))}{|x-y|^{n+s\frac{n}{s}}}\, dx\, dy$$

$$III := \int_{B\backslash B_{\gamma\rho}}\int_{B_{2\rho^{\kappa_2}}} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\ (\psi(x)-\psi(y))}{|x-y|^{n+s\frac{n}{s}}}\, dx\, dy$$

and

$$IV := \int_B\int_B \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\ ((I^{t_0}\varphi - \psi)(x) - (I^{t_0}\varphi - \psi)(y))}{|x-y|^{n+s\frac{n}{s}}}\, dx\, dy$$

With (6.6), $\operatorname{supp}\phi \subset B_{2\rho^{\kappa_3}} \subset B_{2\rho}$, and (6.3),

$$|I| \precsim \rho^\sigma.$$

With (6.2), (6.7) (for ρ small enough),

$$|II| \precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(B_{\gamma\rho})}[\psi - I^{t_0}\phi]_{W^{s,\frac{n}{s}}(B_{\gamma\rho})} \precsim \rho^{\sigma(\frac{n}{s}-1)}\ \rho^{-(\kappa_3-1)}\rho^{(\kappa_2-\kappa_3)n}.$$

With the disjoint support of the integrals, Hölder inequality ($\frac{n}{t_0} > \frac{n}{s}$), and (6.4),

$$|III| \precsim [u]^{p-1}_{W^{s,\frac{n}{s}}(B)}\ \rho^{t_0-s}\ \rho^{\kappa_2(s-t_0)}\ [\psi]_{W^{t_0,\frac{n}{t_0}}(B)} \precsim \rho^{(\kappa_2-1)(s-t_0)}.$$

Lastly, with (6.5)

$$|IV| \precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(B)}[I^{t_0}\varphi - \psi]_{W^{s,\frac{n}{s}}(B)} \precsim \rho^{(\kappa_1-\kappa_2)(n-t_0)}.$$

If we choose $\kappa_1 = \kappa_2 = \kappa_3 = 1$, we obtain

$$\|T_{t_0,B}u\|_{\frac{n}{n-t_0},B_\rho} \precsim 1,$$

whenever $B_{2\gamma\rho} \subset B$, In particular

$$\|T_{t_0,B}u\|_{\frac{n}{n-t_0},\frac{1}{2\gamma}B} \precsim 1. \tag{6.8}$$

On the other hand, we may take

$$\kappa_1 > \kappa_2 > \kappa_3 = 1.$$

Then we have shown that

$$\|T_{t_0,B}u\|_{\frac{n}{n-t_0},B_{\rho^{\kappa_1}}} \precsim \rho^{\tilde{\sigma}},$$

which holds whenever $B_{\gamma\rho} \subset B$. Equivalently, for an even smaller $\tilde{\sigma}$,

$$\|T_{t_0,B}u\|_{\frac{n}{n-t_0},B_{\rho}} \precsim \rho^{\tilde{\sigma}},$$

which holds whenever $B_{\gamma\rho^{\frac{1}{\kappa_1}}} \subset B$. With (6.8) this estimate also holds whenever $B_{2\gamma\rho} \subset B$, with a constant depending on the radius of B. $\square$

In [21] it is shown that for $t_1 > t_0$, $T_{t_1,B}u = I^{t_1-t_0}T_{t_0,B}u$. Since according to Proposition 6.2 $T_{t_0,B}u$ belongs to a Morrey space, we can apply Adams estimates on Riesz potential acting on Morrey spaces [1, Theorem 3.1 and Corollary after Proposition 3.4] and obtain an increased integrability estimate for $T_{t_1,B}u$.

Proposition 6.3. *Under the assumptions of Theorem 6.1 there are $\gamma > 1$, $t_0 < t_1 < s$, and $p_1 > \frac{n}{n-t_1}$ so that*

$$\|T_{t_1,B}u\|_{p_1,B_\rho} \leq C_\Lambda \rho^{\sigma}$$

for any ball so that $B_{\gamma\rho} \subset B$.

Now we exploit (6.1): For any $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$(-\Delta)^s_{\frac{n}{s},B}u[\varphi] = \int_{\mathbb{R}^n} T_{t_1,B}u\ (-\Delta)^{\frac{t_1}{2}}\varphi.$$

Let $\varphi \in C_c^\infty(B_{\frac{1}{4}\rho})$ for $B_{\gamma\rho} \subset B$. With the usual cutoff-function $\eta \in C_c^\infty(B_\rho)$, $\eta \equiv 1$ on $B_{\frac{1}{2}\rho}$

$$|(-\Delta)^s_{\frac{n}{s},B}u[\varphi]| \precsim \|T_{t_1,B}u\|_{p_1,B_\rho}\|(-\Delta)^{\frac{t}{2}}\varphi\|_{p_1',B_\rho} + \|T_{t_1,B}u\|_{\frac{n}{n-t},B_\rho}\|(-\Delta)^{\frac{t_1}{2}}\varphi\|_{\frac{n}{t},\mathbb{R}^n\setminus B_{\frac{1}{2}\rho}}.$$

By the Sobolev inequality for Gagliardo-Norms [21, Theorem 1.6], and the disjoint support [3, Lemma A.1], this implies

$$|(-\Delta)^s_{\frac{n}{s},B}u[\varphi]| \precsim C_\Lambda [\varphi]_{W^{s+t_1-\frac{n}{p_1'},\frac{n}{s}}(\mathbb{R}^n)}.$$

Since $p_1 > \frac{n}{n-t_1}$, we have $s + t_1 - \frac{n}{p_1'} < s$, and the claim of Theorem 1.7 follows from Theorem 1.3 by a covering argument. $\square$

7. Compactness for $\frac{n}{s}$-harmonic maps: Proof of Theorem 1.8

From the arguments in [6, Proof of Lemma 2.3.] one has the following:

Proposition 7.1. *For $s \in (0,1)$, $p \in (1,\infty)$ let $(u_k)_{k=1}^{\infty} \in W^{s,p}(\mathbb{R}^n, \mathbb{S}^{N-1})$, $\Lambda := \sup_{k\in\mathbb{N}} [u_k]_{W^{s,p}(\mathbb{R}^n)} < \infty$ and $\varepsilon_0 > 0$ given. Then up to a subsequence there is $u_\infty \in \dot{W}^{s,p}(\mathbb{R}^n, \mathbb{S}^{N-1})$ and a finite set of points $J = \{a_1, \ldots, a_l\}$ such that*

$$u_k \rightharpoonup u_\infty \quad \text{in } W^{s,p}(\mathbb{R}^n, \mathbb{S}^{N-1}) \text{ as } k \to \infty,$$

and for all $x \notin J$ there is $r = r_x > 0$ so that

$$\limsup_{k\to\infty} [u_k]_{W^{s,p}(B_r(x))} < \varepsilon_0.$$

This, Theorem 1.7 and the compactness of the embedding $W^{s+\delta,\frac{n}{s}}(B_r(x)) \hookrightarrow W^{s,\frac{n}{s}}(B_r(x))$ immediately implies that

$$u_k \xrightarrow{k\to\infty} u_\infty \quad \text{in } W^{s,\frac{n}{s}}_{loc}(\mathbb{R}^n \backslash J).$$

$\square$.

Appendix A. Useful Tools

The following Lemma is used to restrict the fractional p-Laplacian to smaller sets.

Lemma A.1 (Localization Lemma)**.** *Let $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \Omega \subset \mathbb{R}^n$ be open sets so that* $\operatorname{dist}(\Omega_1, \Omega_2^c), \operatorname{dist}(\Omega_2, \Omega_3^c), \operatorname{dist}(\Omega_3, \Omega^c) > 0$. *Let $s \in (0,1)$, $p \in [2,\infty)$.*

For any $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ so that

1. $\tilde{u} - u \equiv const$ *in* Ω_1
2. $\operatorname{supp} \tilde{u} \subset \Omega_2$
3. $[\tilde{u}]_{W^{s,p}(\mathbb{R}^n)} \precsim [u]_{W^{s,p}(\Omega)}$
4. *For any $t \in (2s-1, s)$,*

$$\|(-\Delta)^s_{p,\Omega_3} \tilde{u}\|_{(W^{t,p}_0(\Omega_3))^*} \precsim \|(-\Delta)^s_{p,\Omega} u\|_{(W^{t,p}_0(\Omega))^*} + [u]^{p-1}_{W^{s,p}(\Omega)}.$$

The constants are uniform in u and depend only on s, t, p and the sets Ω_1, Ω_2, Ω_3, and Ω.

Proof. Let $\Omega_1 \Subset \Omega$, let $\eta \equiv \eta_{\Omega_1} \in C_c^\infty(\Omega_2)$, $\eta_{\Omega_1} \equiv 1$ on Ω_1. We set

$$\tilde{u} := \eta_{\Omega_1}(u - (u)_{\Omega_1}).$$

Clearly $\tilde{u}$ satisfies property (1) and (2). We have property (3), too:

$$[\tilde{u}]_{W^{s,p}(\mathbb{R}^n)} \precsim [u]_{W^{s,p}(\Omega)}.$$

We write

$$\tilde{u}(x) - \tilde{u}(y) = \underbrace{\eta(x)(u(x) - u(y))}_{a(x,y)} + \underbrace{(\eta(x) - \eta(y))(u(y) - (u)_{\Omega_1})}_{b(x,y)}.$$

Setting

$$T(a) := |a|^{p-2}a,$$

observe that

$$|T(a+b) - T(a)| \precsim |b| \left(|a|^{p-2} + |b|^{p-2}\right).$$

Also note that

$$T(a(x,y)) = \eta^{p-1}(x)|u(x) - u(y)|^{p-2}(u(x) - u(y))$$

We thus have for any $\varphi \in C_c^\infty(\Omega_3)$,

$$\begin{aligned}
&(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi]\\
=&\int_\Omega\int_\Omega \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-2}(\tilde{u}(x) - \tilde{u}(y))\ (\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\ dx\ dy\\
=&\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\ \eta^{p-1}(x)\ (\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\ dx\ dy\\
&+\int_\Omega\int_\Omega \frac{(T(a+b) - T(a))\ (\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\ dx\ dy\\
=&\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\ (\eta^{p-1}(x)\varphi(x) - \eta^{p-1}(y)\varphi(y))}{|x-y|^{n+sp}}\ dx\ dy\\
&-\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\ (\eta^{p-1}(x) - \eta^{p-1}(y))\varphi(y)}{|x-y|^{n+sp}}\ dx\ dy\\
&+\int_\Omega\int_\Omega \frac{(T(a+b) - T(a))\ (\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\ dx\ dy\\
=&(-\Delta)^s_{p,\Omega}u[\eta^{p-1}\ \varphi]\\
&-\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\ (\eta^{p-1}(x) - \eta^{p-1}(y))\varphi(y)}{|x-y|^{n+sp}}\ dx\ dy\\
&+\int_\Omega\int_\Omega \frac{(T(a+b) - T(a))\ (\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\ dx\ dy.
\end{aligned}$$

So we have that

$$
\begin{aligned}
&|(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi]|\\
\precsim&\|(-\Delta)^s_{p,\Omega}u\|_{(W^{t,p}_0(\Omega))^\ast}\ [\eta^{p-1}\varphi]_{W^{t,p}(\Omega)}\\
&+\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-1}\ |\eta^{p-1}(x)-\eta^{p-1}(y)|\ |\varphi(y)|}{|x-y|^{n+sp}}\ dx\ dy\\
&+\int_\Omega\int_\Omega \frac{|\eta(x)-\eta(y)|\ |u(y)-(u)_{\Omega_1}|\ \eta(x)^{p-2}\ |u(x)-u(y)|^{p-2}\ |\varphi(x)-\varphi(y)|}{|x-y|^{n+sp}}\ dx\ dy\\
&+\int_\Omega\int_\Omega \frac{|\eta(x)-\eta(y)|^{p-1}\ |u(y)-(u)_{\Omega_1}|^{p-1}\ |\varphi(x)-\varphi(y)|}{|x-y|^{n+sp}}\ dx\ dy.
\end{aligned}
$$

That is for any $t<s$

$$
\begin{aligned}
&|(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi]|\\
\precsim&\|(-\Delta)^s_{p,\Omega}u\|_{(W^{t,p}_0(\Omega))^\ast}\ [\eta^{p-1}\varphi]_{W^{t,p}(\Omega)}\\
&+[u]^{p-1}_{W^{s,p}(\Omega)}\ \left(\int_\Omega\int_\Omega \frac{|\eta^{p-1}(x)-\eta^{p-1}(y)|^p\ |\varphi(y)|^p}{|x-y|^{n+sp}}\ dx\ dy\right)^{\frac{1}{p}}\\
&+[\varphi]_{W^{t,p}(\Omega)}\ [u]^{p-2}_{W^{s,p}(\Omega)}\left(\int_\Omega\int_{\Omega_2} \frac{|\eta(x)-\eta(y)|^p\ |u(y)-(u)_{\Omega_1}|^p}{|x-y|^{n+(2s-t)p}}\ dx\ dy\right)^{\frac{1}{p}}\\
&+[\varphi]_{W^{t,p}(\Omega)}\left(\int_\Omega\int_{\Omega_2} \frac{|\eta(x)-\eta(y)|^p\ |u(y)-(u)_{\Omega_1}|^p}{|x-y|^{n+(2s-t)p}}\ dx\ dy\right)^{\frac{p-1}{p}}.
\end{aligned}
$$

Since η is bounded and Lipschitz, $\operatorname{supp}\eta\subset\Omega_2$, and $\varphi\in C^\infty_c(\Omega_3)$ we have that

$$
[\eta^{p-1}\varphi]_{W^{t,p}(\Omega)}\precsim[\varphi]_{W^{t,p}(\mathbb{R}^n)}.
$$

Also, choosing some bounded $\Omega_4\Subset\Omega$ so that $\Omega_3\Subset\Omega_4$,

$$
\begin{aligned}
&\int_\Omega\int_\Omega \frac{|\eta^{p-1}(x)-\eta^{p-1}(y)|^p\ |\varphi(y)|^p}{|x-y|^{n+sp}}\ dx\ dy\\
\precsim&\int_{\Omega_3}\int_{\Omega_4}|x-y|^{(1-s)p-n}\ \ dx\ |\varphi(y)|^p dy\\
&+\int_{\Omega_3}\int_{\mathbb{R}^n\backslash\Omega_4}|x-y|^{-n-sp}\ \ dx\ |\varphi(y)|^p dy\\
\precsim&\|\varphi\|_p^p\precsim[\varphi]^p_{W^{t,p}(\mathbb{R}^n)}.
\end{aligned}
$$

Finally, using Lipschitz continuity of η and that $2s-1<t<s$

$$
\begin{aligned}
&\int_\Omega\int_{\Omega_2} \frac{|\eta(x)-\eta(y)|^p\ |u(y)-(u)_{\Omega_1}|^p}{|x-y|^{n+(2s-t)p}}\ dx\ dy\\
\precsim &\int_{\Omega_3} |u(y)-(u)_{\Omega_1}|^p \int_{\Omega_2} |x-y|^{-n+(t+1-2s)p}\ dx\ dy\\
&+\int_{\Omega\backslash\Omega_3} |u(y)-(u)_{\Omega_1}|^p \int_{\Omega_2} \frac{1}{|x-y|^{n+sp}}\ dx\ dy\\
\precsim &\int_{\Omega_1}\int_{\Omega_3} |u(y)-u(z)|^p\ dy\ dz\\
&+\int_{\Omega_1}\int_{\Omega\backslash\Omega_3} |u(y)-u(z)|^p \int_{\Omega_2} \frac{1}{|x-y|^{n+sp}}\ dx\ dy\ dz
\end{aligned}
$$

Note that for $x,z\in\Omega_2$ and $y\in\Omega_3^c$ we have that $|x-y|\approx|y-z|$, and since $\Omega_1,\Omega_2,\Omega_3$ are bounded we then have

$$
\int_\Omega\int_{\Omega_2} \frac{|\eta(x)-\eta(y)|^p\ |u(y)-(u)_{\Omega_1}|^p}{|x-y|^{n+(2s-t)p}}\ dx\ dy \precsim [u]_{W^{s,p}(\Omega)}
$$

Thus we have shown that for any $\varphi\in C_c^\infty(\Omega_3)$,

$$
|(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi]| \precsim \left(\|(-\Delta)^s_{p,\Omega}u\|_{(W_0^{t,p}(\Omega))^*} + [u]^{p-1}_{W^{s,p}(\Omega)}\right)\ [\varphi]_{W^{t,p}(\mathbb{R}^n)}.
$$

Since moreover, $\operatorname{supp}\tilde{u}\subset\Omega_2$, for any $\varphi\in C_c^\infty(\Omega_3)$,

$$
|(-\Delta)^s_{p,\Omega_3}\tilde{u}[\varphi]| \precsim |(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi]| + [u]^{p-1}_{W^{s,p}(\Omega)}\ [\varphi]_{W^{t,p}(\mathbb{R}^n)},
$$

we get the claim. □

The next Lemma estimates the $W^{s,p}$-norm in terms of the fractional p-Laplacian.

Lemma A.2. *Let $B\subset\mathbb{R}^n$ be a ball and $4B$ the concentric ball with four times the radius. Then for any $\delta>0$, $[u]^p_{W^{s,p}(B)}$ can be estimated by*

$$
\begin{aligned}
&\delta^p[u]^p_{W^{s,p}(4B)}\\
&+\frac{C}{\delta^{p'}}\left(\sup_\varphi \int_{4B}\int_{4B} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\ (\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\ dx\ dy\right)^{\frac{p}{p-1}}\\
&+\frac{C}{\delta^{p'}}\ \operatorname{diam}(B)^{-sp} \int\limits_{4B} |u(x)-(u)_B|^p\ dx
\end{aligned}
$$

where the supremum is over all $\varphi\in C_c^\infty(2B)$ and $[\varphi]_{W^{s,p}(\mathbb{R}^n)}\leq 1$.

Proof. Let $\eta \in C_c^\infty(2B)$, $\eta \equiv 1$ in B be the usual cutoff function in $2B$.

$$\psi(x) := \eta(x)(u(x) - (u)_B), \quad \text{and} \quad \varphi(x) := \eta^2(x)(u(x) - (u)_B).$$

Then,

$$[\psi]_{W^{s,p}(\mathbb{R}^n)} + [\varphi]_{W^{s,p}(\mathbb{R}^n)} \precsim [u]_{W^{s,p}(2B)}. \tag{A.1}$$

We have

$$[u]^p_{W^{s,p}(B)} \leq \int\limits_{4B}\int\limits_{4B} \frac{|u(x)-u(y)|^{p-2}(\psi(x)-\psi(y))\ (\psi(x)-\psi(y))}{|x-y|^{n+sp}} dx\ dy$$

Now we observe

$$\begin{aligned}(\psi(x)-\psi(y))^2 =&(\psi(x)-\psi(y))(\eta(x)-\eta(y))(u(x)-(u)_B)\\ &+\psi(x)(\eta(y)-\eta(x))\ (u(x)-u(y))\\ &+(\varphi(x)-\varphi(y))(u(x)-u(y)).\end{aligned}$$

That is,

$$[u]^p_{W^{s,p}(B)} \precsim I + II + III,$$

with

$$I := \int\limits_{4B}\int\limits_{4B} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} dx\ dy,$$

$$II := \int\limits_{4B}\int\limits_{4B} \frac{|u(x)-u(y)|^{p-2}|\eta(x)-\eta(y)|\ |\psi(x)-\psi(y)|}{|x-y|^{n+sp}}\ |u(x)-(u)_B|\ dx\ dy,$$

$$III := \int\limits_{4B}\int\limits_{4B} \frac{|u(x)-u(y)|^{p-1}|\eta(x)-\eta(y)|}{|x-y|^{n+sp}}\ |\psi(x)|dx\ dy.$$

With (A.1),

$$I \leq [u]_{W^{s,p}(4B)} \sup_{[\varphi]_{W^{s,p}(\mathbb{R}^n)}\leq 1} \int\limits_{4B}\int\limits_{4B} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\ (\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\ dx\ dy.$$

As for II,

$$II \precsim \|\nabla\eta\|_\infty \int\limits_{4B}\int\limits_{4B} \frac{|u(x)-u(y)|^{p-2}|\psi(x)-\psi(y)|\ |u(x)-(u)_B|}{|x-y|^{n+sp-1}}\ dx\ dy.$$

For any $t_2 > 0$ so that $t_2 = 1 - s$, we have with Hölder's inequality

$$II \precsim \|\nabla\eta\|_\infty \int\limits_{4B}\int\limits_{4B} \frac{|u(x)-u(y)|^{p-2}|\psi(x)-\psi(y)|\ |u(x)-(u)_B|}{|x-y|^{n+s(p-2)+s-t_2}}\ dx\ dy$$

$$\precsim \operatorname{diam}(B)^{-1}\ [u]^{p-2}_{W^{s,p}(4B)}\ [\psi]_{W^{s,p}(4B)}\ \left(\int\limits_{4B}\int\limits_{4B} \frac{|u(x)-(u)_B|^p}{|x-y|^{n-t_2p}}\ dx\ dy\right)^{\frac{1}{p}}.$$

Since $t_2 > 0$,

$$\int\limits_{4B}\int\limits_{4B} \frac{|u(x)-(u)_B|^p}{|x-y|^{n-t_2p}}\ dx\ dy \precsim (\operatorname{diam} B)^{t_2p} \int\limits_{4B} |u(x)-(u)_B|^p\ dx$$

So using again (A.1), we arrive at

$$II \precsim \operatorname{diam}(B)^{-s}\ [u]^{p-1}_{W^{s,p}(4B)}\ \left(\int\limits_{4B} |u(x)-(u)_B|^p\ dx\right)^{\frac{1}{p}}.$$

III can be estimated the same way as II, and we have the following estimate for $[u]^p_{W^{s,p}(B)}$

$$[u]_{W^{s,p}(4B)}\ \sup_\varphi \int\limits_{4B}\int\limits_{4B} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\ (\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\ dx\ dy$$

$$+ [u]^{p-1}_{W^{s,p}(4B)}\ \operatorname{diam}(B)^{-s}\ \left(\int\limits_{4B} |u(x)-(u)_B|^p\ dx\right)^{\frac{1}{p}}$$

We conclude with Young's inequality. □

The next Proposition follows immediately from Jensen's inequality and the definition of $[u]^p_{W^{t,p}(\lambda B)}$.

Proposition A.3 (A Poincaré type inequality)**.** *Let B be a ball and for $\lambda \geq 1$ let λB be the concentric ball with λ times the radius. Then for any $t \in (0,1)$, $p \in (1,\infty)$,*

$$\int\limits_{\lambda B} |u(x)-(u)_B|^p\ dx \precsim \lambda^{n+tp}\operatorname{diam}(B)^{tp}\ [u]^p_{W^{t,p}(\lambda B)}.$$

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